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Direct Sums of Countable Groups* and Related Concepts†

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1. Introduction

Direct sums of countable groups have many features in common with direct sums of cyclic groups, the two concepts coinciding for torsion groups without elements of infinite height. Moreover, they have a sufficiently rich structure to warrant investigation without being complicated enough to defy analysis. The purpose of this paper is to investigate the various properties of these groups and their interrelations.

The main result of Section 2 answers in the affirmative a question raised by Cutler [1]: If G/G^1 is a direct sum of cyclic groups and G^1 is a direct sum of countable groups, is G a direct sum of countable groups? ($G^1 = \bigcap_n nG$).

Section 3 is devoted to subsidiary ideas related to direct sums of countable groups. Nunke [5] has shown that a subgroup of a direct sum of countable reduced p-groups need not be a direct sum of countable groups. We show that in fact such a group can fail to be in the (perhaps) larger class of groups described in problem five of Fuchs [2]; i.e., those groups in which every infinite subgroup can be imbedded in a summand of the same cardinality. To this end we introduce the concept of a Q-group which, for groups without elements of infinite height, is (perhaps) weaker than "Fuchs 5."

2. A Sufficient Condition for a Group to be a Direct Sum of Countable Groups

If G is a direct sum of countable primary groups then G^1 is a direct sum of countable groups and G/G^1 is a direct sum of cyclic groups. In this section

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we show that the converse statement is true for arbitrary groups and thus acquire a powerful tool for showing a group is a direct sum of countable groups.

Let $| G |$ = cardinality of G.

Notation. If S is a subgroup of $\bigoplus_{i\in I} P_i$ then $S_c = \bigoplus_{i\in J} P_i$ where J is the least subset of I such that $S \subset \bigoplus_{i\in J} P_i$. Note that if $| P_i | \leqslant \aleph_0$ for all $i \in I$ and $| S | \geqslant \aleph_0$ then $| S_c | = | S |$. Note also that this definition depends on the given decomposition of $\bigoplus P_i$. Throughout a given argument we shall assume a fixed decomposition.

If S is an infinite subgroup of G then T is a *purification of* S if T is a pure subgroup of G containing S and $| T | = | S |$.

LEMMA. *Let G be an abelian group such that G^1 is a direct sum of countable groups and G/G^1 is a direct sum of cyclic groups. Let f be the natural map from G onto G/G^1. If A is an infinite subgroup of G then there exists a purification H of A satisfying*:

(i) $f(H)_c = f(H)$

(ii) $(H \cap G^1)_c = H \cap G^1$.

Furthermore, any pure subgroup H of G satisfying (i) *and* (ii) *is a summand of G.*

Proof. We construct increasing sequences of groups $K_n \subset G^1$, $C_n \subset G/G^1$, and $H_n \subset G$, $n = 1, 2, 3, \ldots$, as follows:

$$\begin{aligned} H_0 = A,\ K_n &= (H_{n-1} \cap G^1)_c \\ C_n &= f(H_{n-1})_c \\ H_n &= \text{purification of } (K_n + S_n + H_{n-1}) \text{ where } S_n \text{ is such that} \\ &\quad f(S_n) = C_n \text{ and } | S_n | \leqslant \aleph_0 \, | C_n |. \end{aligned}$$

Set $H = \cup H_n$, $K = \cup K_n$, and $C = \cup C_n$. Clearly H is pure, $K = K_c$ and $C = C_c$. Moreover $f(H) = C$ (since $f(H_{n-1}) \subset C_n \subset f(H_n)$) and $H \cap G^1 = K$ (since $H_{n-1} \cap G^1 \subset K_n \subset H_n \cap G^1$). Also $| H | = | A |$ and $A \subset H$.

Now let H be any pure subgroup of G satisfying (i) and (ii). We shall show that H is a summand of G. Write $G^1 = K \oplus K^*$, $G/G^1 = C \oplus C^*$, where $C = f(H)$ and $K = H \cap G^1$. Let $\{x_i\}$ be a set of independent generators of C^*. Choose $y_i \in G$ such that $f(y_i) = x_i$ and let m_i be the order of x_i. Then $m_i y_i = k_i + k_i^* \in K \oplus K^* = G^1$. Since H is pure, $H^1 = H \cap G^1 = K$ and so there is an $h_i \in H$ such that $m_i h_i = k_i$. Set $z_i = y_i - h_i$ and let V be generated by K^* and the z_i's. Claim: $-V \cap H = 0$ and $V + H = G$. Suppose $v \in V \cap H$. Then $v = \sum s_i z_i + k^*$, $k^* \in K^*$. Now $f(v) = \sum s_i x_i - \sum s_i f(h_i) \in C$, since $v \in H$. Therefore $\sum s_i x_i \in C$ and so $\sum s_i x_i = 0$.

Therefore $s_i = t_i m_i$ and so $v = \sum t_i m_i z_i + k^*$. But $m_i z_i = k_i^* \in K^*$ and so $v \in K^*$. But $K^* \cap H = 0$ and hence $v = 0$. Clearly $f(V) + f(H) = G/G^1$ and $V + H \supset G^1$ and so $V + H = G$.

THEOREM 1. *If G is an abelian group such that G^1 is a direct sum of countable groups and G/G^1 is a direct sum of cyclic groups, then G is a direct sum of countable groups.*

Proof. We shall show that if $|G| = \alpha > \aleph_0$, then G is a direct sum of groups of cardinality strictly less than α. Since the hypotheses are inherited by summands the theorem follows by induction.

If A is an infinite subgroup of G we denote by $\bar{A}$ a purification of A satisfying (i) and (ii) of the lemma.

Let $\{g_j\}$ be a well ordering of the elements of G by the ordinals $j < \alpha$. Construct H_j inductively by:

$$H_0 = 0 \qquad H_{j+1} = \overline{(H_j + g_j)}$$

$$H_j = \bigcup_{i<j} H_i \qquad \text{if } j \text{ is a limit ordinal}$$

Clearly the H_j's satisfy (i) and (ii) of the lemma and are pure. Therefore they are summands of G. Moreover $|H_j| < \alpha$ and $\bigcup_{j<\alpha} H_j = G$. Let $P_j \oplus H_j = H_{j+1}$. $G = \bigoplus_{j<\alpha} P_j$ is the desired decomposition.

COROLLARY 1. *Let G be a p-group of finite Ulm type n which is a direct sum of countable groups. If H is a subgroup of G and $p^\alpha H = H \cap p^\alpha G$ for all $\alpha < \omega(n-1)$ then H is a direct sum of countable groups.*

Proof. Induction on n. If $n = 1$, G is a direct sum of cyclic groups and the assertion follows immediately. If $n > 1$ then H is pure in G and so $H/H^1 \subset G/G^1$ is a direct sum of cyclic groups. Moreover, if $\alpha < \omega(n-2)$, $p^\alpha H^1 = p^{\omega+\alpha} H = H \cap p^{\omega+\alpha} G = H \cap p^\alpha G^1 = H^1 \cap p^\alpha G^1$. Therefore, since the Ulm type of G^1 is $n-1$, H^1 is a direct sum of countable groups. By the Theorem, H is a direct sum of countable groups.

COROLLARY 2. *If G is a direct sum of countable p-groups and G^1 is a direct sum of cyclic groups then any pure subgroup of G is a direct sum of countable groups.*

The following answers a question of Cutler [1].

COROLLARY 3. *If G is a direct sum of countable p-groups and $p^n G \subseteq H \subseteq G$ then H is a direct sum of countable groups.*

Proof. Notice that $p^n H \subseteq p^n G \subseteq H$ so $H^1 = G^1$ is a direct sum of countable groups, and $H/H^1 \subseteq G/G^1$ is a direct sum of cyclic groups.

3. FULLY STARRED, FUCHS 5 AND Q-GROUPS

Direct sums of countable groups enjoy many properties which when considered alone yield interesting classes of groups. The first property we will consider is due to Khabbaz [3]. A group G is called *starred* if whenever G/H is divisible, H a subgroup of G, then $|H| = |G|$. A group is *fully starred* if every subgroup is starred. Subgroups of direct sums of reduced countable groups are easily seen to be fully starred. Infinite starred p-groups have been characterized in [3] by the equality $|G/pG| = |G|$. The following proposition is often useful for showing a p-group to be (fully) starred.

PROPOSITION 1. *If $0 \to A \to B \to C \to 0$ is an exact sequence of p-groups and C and A are starred, then B is starred.*

Proof. Tensoring Z_p with the exact sequence $0 \to A \to B \to C \to 0$ yields the exact sequence $C[p] \xrightarrow{g} A/pA \xrightarrow{f} B/pB \to C/pC \to 0$. Hence $|B/pB| \geqslant |C/pC| = |C|$ and so if $|B| = |C|$, we are done. If $|C| < |B|$ then $|A/pA| = |A| = |B|$. But

$$|\ker(f)| = |\operatorname{im}(g)| \leqslant |C[p]| = |C| < |B| = |A/pA|.$$

Therefore $|\operatorname{im}(f)| = |A/pA| = |B|$ and so $|B/pB| = |B|$.

COROLLARY 1. *If $0 \to A \to B \to C \to 0$ is an exact sequence of p-groups and C and A are fully starred then B is fully starred.*

COROLLARY 2. *If $C = B/A$ and C and A are direct sums of cyclic p-groups then B is fully starred.*

The second property was stated by Fuchs [2] in his fifth unsolved problem. A group G is said to be *Fuchs 5* if every infinite subgroup H can be imbedded in a summand T of the same cardinality. It is easily seen that direct sums of countable groups are Fuchs 5. The converse is still open. Not much is known about Fuchs 5 groups. The interested reader is referred to [6]. We summarize some of the known results in the following.

PROPOSITION 2. *If B is a fully invariant subgroup of A and A is Fuchs 5 then B and A/B are Fuchs 5.*

Proof. Let H be an infinite subgroup of B. Then $A = T \oplus V$, $T \supseteq H$, $|T| = |H|$. But $B = T \cap B \oplus V \cap B$ and $T \cap B \supseteq H$, $|T \cap B| = |H|$.

Let H be an infinite subgroup of A/B. We can find a subgroup K of A such that $|K| = |H|$ and $(K + B)/B = H$. Then $A = T \oplus V$, $T \supseteq K$, $|T| = |K| = |H|$. But $B = T \cap B \oplus V \cap B$ and so

$$T/T \cap B \oplus V/V \cap B \cong A/B = (T + B)/B \oplus (V + B)/B$$

and $(T + B)/B \supseteq H$, $|(T + B)/B| = |H|$.

COROLLARY. *If G is Fuchs 5 so are $p^\alpha G$, $G/p^\alpha G$, G_t, G/G_t, $G[n]$, divisible part of G, reduced part of G, G^1, and G/G^1.*

There is little difficulty in showing that any reduced Fuchs 5 group is fully starred.

We can find an example of a fully starred group which is not Fuchs 5 even among groups without elements of infinite height. For groups without elements of infinite height it is convenient to consider a concept between Fuchs 5 and fully starred.

DEFINITION. A group G without elements of infinite height is called a *Q-group* if for every infinite subgroup H of G, $|(G/H)^1| \leqslant |H|$.

Q-groups form a natural subclass of fully starred groups in the topological setting. A group G is fully starred if no subgroup H contains a dense set of smaller cardinality in the topology of H. A group G is a Q-group if no subgroup H contains a dense subset of smaller cardinality in the relative (G) topology on H. (The topology in question is the n-adic; i.e. $\{nG\}$ forms a neighborhood system for 0.)

We list a few facts about Q-groups.

PROPOSITION 3.

1. *If G is Fuchs 5 and $G^1 = 0$ then G is a Q-group.*
2. *If G is a Q-group then G is fully starred.*
3. *If G is a Q-group then every subgroup of G is a Q-group.*
4. *If G_i, $i \in I$, is a set of Q-groups then $G = \oplus_{i \in I} G_i$ is a Q-group.*
5. *If G is a group and $n \neq 0$ is an integer then G is a Q-group $\Leftrightarrow$ nG is a Q-group.*

Proof. 1. Let H be an infinite subgroup of G. Let T be a summand of G containing H and of the same cardinality as H. Then $G/H \cong T/H \oplus A$

where $A^1 = 0$ and so $(G/H)^1 \simeq (T/H)^1$ which has cardinality no greater than $|H|$.

2. By (3) it suffices to show that G is starred. Let H be an infinite subgroup of G such that G/H is divisible. Then $(G/H)^1 = G/H$ and so

$$|H| = |(G/H)^1| \cdot |H| = |G|.$$

Therefore $|H| = |G|$.

3. Let K be a subgroup of G and H an infinite subgroup of K. $K/H \subseteq G/H$ and so $(K/H)^1 \subseteq (G/H)^1$. But $|(G/H)^1| \leqslant |H|$ and so, a fortiori,

$$|(K/H)^1| \leqslant |H|.$$

4. Let H be an infinite subgroup of G. We may assume that $|I| \leqslant |H|$. Let H_i be the projection of H in G_i and consider $H^* = \bigoplus_{i \in I} H_i$. Note that $|H^*| = |H|$. We have the exact sequence $0 \to H^*/H \to G/H \to G/H^* \to 0$. Now $G/H^* \simeq \bigoplus_{i \in I} G_i/H_i$ and so $|(G/H^*)^1| \leqslant |H|$. Therefore $|(G/H)^1| \leqslant |H|$.

5. $\Rightarrow$ Follows from (3).

$\Leftarrow$ Let H be an infinite subgroup of G such that $|(G/H)^1| > |H|$. Again we may assume that H is pure. Then $n(G/H) = (nG + H)/H \simeq nG/nH$ and so $|(nG/nH)^1| = |(G/H)^1| > |H| \geqslant |nH|$. nH cannot be finite lest H be a summand.

Before presenting the example alluded to above we need the following bit of homological fluff.

LEMMA. $p^\alpha \operatorname{Tor}(A, B) = \operatorname{Tor}(p^\alpha A, p^\alpha B)$, *$p$ a prime, α an ordinal number.*

Proof. An element of $\operatorname{Tor}(pA, pB)$ has the form $cls\langle pa, n, pb\rangle$, $a \in A$, $b \in B$, n an integer such that $npa = 0 = npb$ (MacLane [4]). But

$$cls\langle pa, n, pb\rangle = p(cls\langle a, pn, b\rangle) \in p \operatorname{Tor}(A, B).$$

Thus we have shown that $\operatorname{Tor}(pA, pB) \subseteq p \operatorname{Tor}(A, B)$. For general α we disintinguish between two cases. If α is a limit ordinal $\operatorname{Tor}(p^\alpha A, p^\alpha B) \subseteq \bigcap_{\beta<\alpha} \operatorname{Tor}(p^\beta A, p^\beta B) = \bigcap_{\beta<\alpha} p^\beta \operatorname{Tor}(A, B) = p^\alpha \operatorname{Tor}(A, B)$ by induction. If $\alpha = \beta + 1$, $\operatorname{Tor}(p^\alpha A, p^\alpha B) = \operatorname{Tor}(pp^\beta A, pp^\beta B) \subseteq p \operatorname{Tor}(p^\beta A, p^\beta B) = p^\alpha \operatorname{Tor}(A, B)$ by induction. It remains to show that

$$p^\alpha \operatorname{Tor}(A, B) \subseteq \operatorname{Tor}(p^\alpha A, p^\alpha B).$$

We first note that if $p^\alpha A = 0$ then $p^\alpha \operatorname{Tor}(A, B) = 0$. To see this imbed B in a divisible group D. Then $\operatorname{Tor}(A, B) \subseteq \operatorname{Tor}(A, D) = \bigoplus A_j$, $(A_j \simeq A)$.

Consider the exact commutative diagram

$$\begin{array}{ccccc}
 & 0 & & 0 & & 0 \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \to & \mathrm{Tor}(p^\alpha A, p^\alpha B) & \to & \mathrm{Tor}(A, p^\alpha B) & \to & \mathrm{Tor}(A/p^\alpha A, p^\alpha B) \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \to & \mathrm{Tor}(p^\alpha A, B) & \to & \mathrm{Tor}(A, B) & \to & \mathrm{Tor}(A/p^\alpha A, B) \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \to & \mathrm{Tor}(p^\alpha A, B/p^\alpha B) & \to & \mathrm{Tor}(A, B/p^\alpha B) & \to & \mathrm{Tor}(A/p^\alpha A, B/p^\alpha B)
\end{array}$$

Elementary diagram chasing shows that the induced sequence

$$0 \to \mathrm{Tor}(p^\alpha A, p^\alpha B) \to \mathrm{Tor}(A, B) \to \mathrm{Tor}(A, B/p^\alpha B) \oplus \mathrm{Tor}(A/p^\alpha A, B)$$

is exact. But $p^\alpha(\mathrm{Tor}(A, B/p^\alpha B) \oplus \mathrm{Tor}(A/p^\alpha A, B)) = 0$. Therefore

$$p^\alpha \,\mathrm{Tor}(A, B) \subseteq \mathrm{Tor}(p^\alpha A, p^\alpha B).$$

COROLLARY. $\mathrm{Tor}(A, B)^1 = \mathrm{Tor}(A^1, B^1)$.

PROPOSITION 4. *Let C be a countable p-group such that $C^1 \neq 0$. A p-group G is a Q-group* $\Leftrightarrow$ $\mathrm{Tor}(G, C)$ *is a Q-group.*

Proof. $\Rightarrow$ Suppose K is an infinite subgroup of $\mathrm{Tor}(G, C)$ and $|(\mathrm{Tor}(G, C)/K)^1| > |K|$. We can find an infinite pure subgroup H of G such that $K \subseteq \mathrm{Tor}(H, C) \subseteq \mathrm{Tor}(G, C)$ and $|K| = |\mathrm{Tor}(H, C)| = |H|$. To see this purify the subgroup generated by representatives in G of K. Consider the exact sequence $0 \to \mathrm{Tor}(H, C) \to \mathrm{Tor}(G, C) \to \mathrm{Tor}(G/H, C) \to 0$. Clearly $|\mathrm{Tor}(G/H, C)^1| > |\mathrm{Tor}(H, C)|$. But $\mathrm{Tor}(G/H, C)^1 = \mathrm{Tor}((G/H)^1, C^1)$ which has cardinality $|(G/H)^1|$.

$\Leftarrow$ Suppose H is an infinite subgroup of G such that $|(G/H)^1| > |H|$. We may assume that H is pure. We then have the exact sequence

$$0 \to \mathrm{Tor}(H, C) \to \mathrm{Tor}(G, C) \to \mathrm{Tor}(G/H, C) \to 0.$$

Now $|\mathrm{Tor}(H, C)| = |H|$ and $\mathrm{Tor}(G/H, C)^1 = \mathrm{Tor}((G/H)^1, C^1)$ so $|\mathrm{Tor}(G/H, C)^1| > |\mathrm{Tor}(H, C)|$ and hence $\mathrm{Tor}(G, C)$ is not a Q-group.

COROLLARY. *There exists a fully starred p-group without elements of infinite height which is a subgroup of a direct sum of countable reduced p-groups but is not a Q-group.*

Proof. $\mathrm{Tor}(A, B)$ is the desired group where A is not fully starred, $A^1 = 0$, and B is countable reduced, $B^1 \neq 0$.

4. A Few Unsettled Questions

The authors feel that the answers to the following questions would be of some interest and importance to the theory of Abelian p-groups:

1. Is every isotype subgroup of a direct sum of countable p-groups also a direct sum of countable groups?
2. Is every Fuchs 5 group a direct sum of countable groups?
3. Is every summand of a Fuchs 5 group also a Fuchs 5 group?
4. Is every Q-group a Fuchs 5 group?
5. Is every Q-group a direct sum of cyclic groups?

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